Lecture 15

In this lecture, we'll prove the Theorem stated in last lecture.

Theorem 1 Let G be a group and $H \triangleleft G$. The set $\frac{G}{H} = \{aH \mid a \in G\}$ is a group under the operation $(aH)\cdot(bH) = abH$.

Proof Since aH can be represented by many elements, we must first make sure that the group operation is well-defined, i.e., if $aH = a'H$ for $a, a' \in G$ and $bH = b'H$ for $b, b' \in G$, then

$$(aH)(bH) = abH = a'b'H = (a'H)(b'H).$$

Now if $aH = a'H \Rightarrow a' = ah_1$ for some $h_1 \in H$

and $b' = bh_2$ for some $h_2 \in H$.

So $a'b'H = ah_1bh_2H = ah_1bH = ah_1Hb$

as $H \triangleleft G$.

So $a'b'H = ah_1Hb = aHb = abH$.

Thus the operation is well-defined.

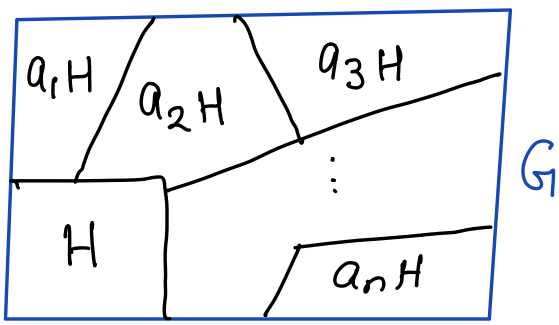
What should be the identity? A natural guess is $eH = H$. So H is the identity in $\frac{G}{H}$.

Inverse of $aH = a^{-1}H$ and associativity follows from associativity in G and $H \triangleleft G$. So $\frac{G}{H}$ is a group.

$\square$

In a way, the group $\frac{G}{H}$ is causing a systematic collapse of elements in G. All the elements in the coset of H containing a collapse to a single element aH in $\frac{G}{H}$.

So the subgroup H becomes the identity in $\frac{G}{H}$. This can be represented by the following schematic (though crude) diagram:-



So H is dividing G into disjoint left cosets $\{H, a_1H, \dots, a_nH\}$ and this is a "smaller" group than G and can give a lot of information about G itself.

<u>Remark</u> The procedure of quotienting out by a part of an object is a very common technique

in Mathematics.

Now that $\frac{G}{H}$ is a group, one can ask that what is it's order. We already know the answer, since $\frac{G}{H}$ is the set of left cosets of H in G, so from Lec. 10, Corollary 1, we have

<u>Theorem 2</u> If G is a finite group and $H \trianglelefteq G$, then $\left|\frac{G}{H}\right| = \frac{|G|}{|H|}$, i.e. the index of H in G.

Let's see some applications of quotient groups as to how information about quotient groups can give us information about the group itself.

<u>Theorem 3</u> $G/Z(G)$ Theorem

Let G be a group and $Z(G)$ be the center of G. If $\frac{G}{Z(G)}$ is cyclic, then G is abelian.

Proof First of all since $Z(G) \trianglelefteq G \Rightarrow \frac{G}{Z(G)}$ makes sense.

Since $\frac{G}{Z(G)}$ is cyclic $\Rightarrow \frac{G}{Z(G)} = \langle aZ(G) \rangle$ for some $a \in G$. —— (*)

We want to show that G is abelian, so let $x, y \in G$ be arbitrary.

To show : $xy = yx$.

Since we have information only about $\frac{G}{Z(G)}$ so it makes sense to look at $xZ(G)$ and $yZ(G)$.

Since they are elements of $\frac{G}{Z(G)}$, so from (*)

$$xZ(G) = a^m Z(G) \quad , \; yZ(G) = a^n Z(G) \; , \; m, n \in \mathbb{Z}.$$

So $x = a^m z_1$ for some $z_1 \in Z(G)$ and

$y = a^n z_2$ for some $z_2 \in Z(G)$

So $xy = a^m z_1 . a^n z_2 = a^m . a^n . z_1 . z_2$ (as $Z(G)$ is the center)

So, $xy = a^n \cdot a^m \cdot z_1 \cdot z_2 = a^n \cdot a^m \cdot z_2 \cdot z_1 = a^n \cdot z_2 \cdot a^m \cdot z_1$

$= yx$

and hence G is abelian.

$\square$

This shows the power of quotient groups, we analysed a "smaller" group $\frac{G}{Z(G)}$ and from that, we gathered information about a larger group G.

In fact if $\frac{G}{Z(G)}$ is cyclic then G is abelian, so $G = Z(G) \Rightarrow \frac{G}{Z(G)}$ is trivial.

Also, if G is non-abelian, then $\frac{G}{Z(G)}$ cannot be cyclic.